PAP1
Experiment 12: "Moments of inertia"

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1 Introduction

1.1 Motivation

In this experiment, we use a rotary pendulum to determine the moment of inertia of a complex shaped brass plate. This will be done by first calculating the directing moment of the pendulum in two different ways and using this information to extrapolate the the moment of inertia from differences in period of the pendulum. Lastly we will try to verify Steiner's Theorem.

1.2 Experiment Setup

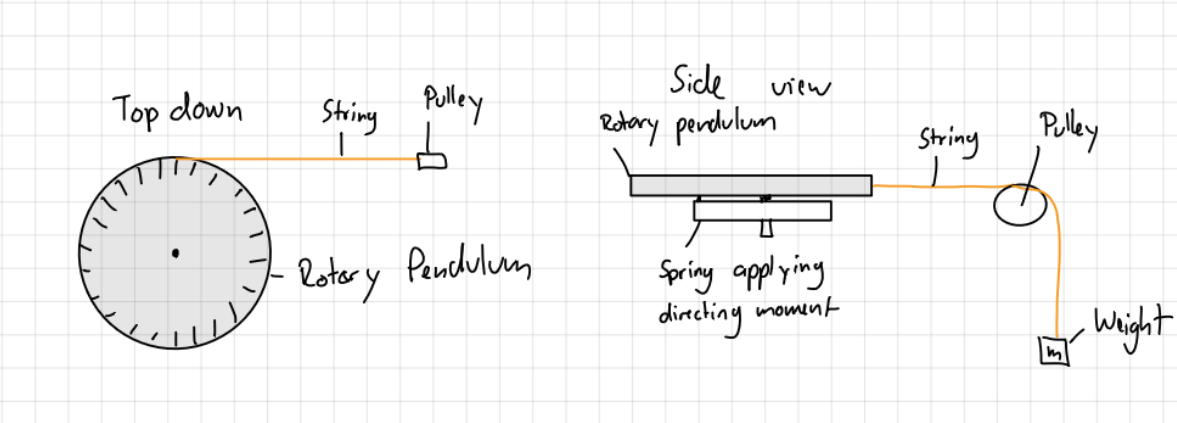


Figure 1: Mesuring the directing moment

For our first measurement we will attach different weights to a string. The opposite end of it will be attached to a rotary pendulum. The string will be redirected via a guide pulley in such a way, that the weight is hanging vertically from it and the string is pulling on the rotary pendulum tangentially. We denote the different angles of deflection for the different weights. Figure 1 shows this.

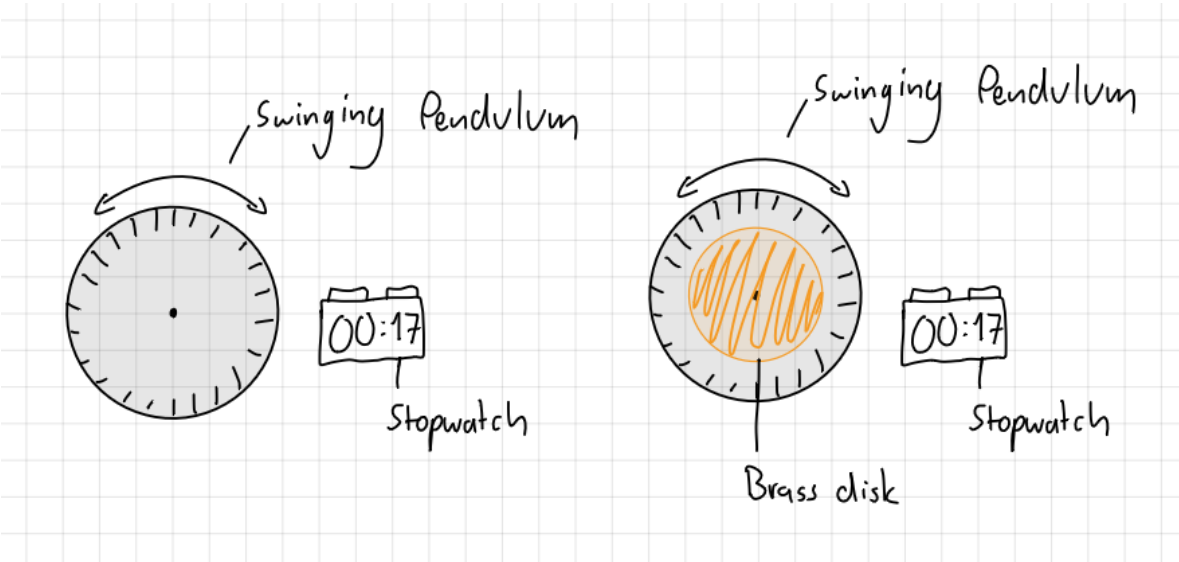


Figure 2: rotary pendulum with and without brass disk

In figure 2 you can see the arrangement for the next line of measurements. We simply displace the

pendulum by an arbitrary angle and measure the time it takes to complete 20 cycles. We do this 3 different times, then place a brass disk on top of the pendulum in such a way, that it is coaxial with the table. We displace the system again and repeat the same measurements.

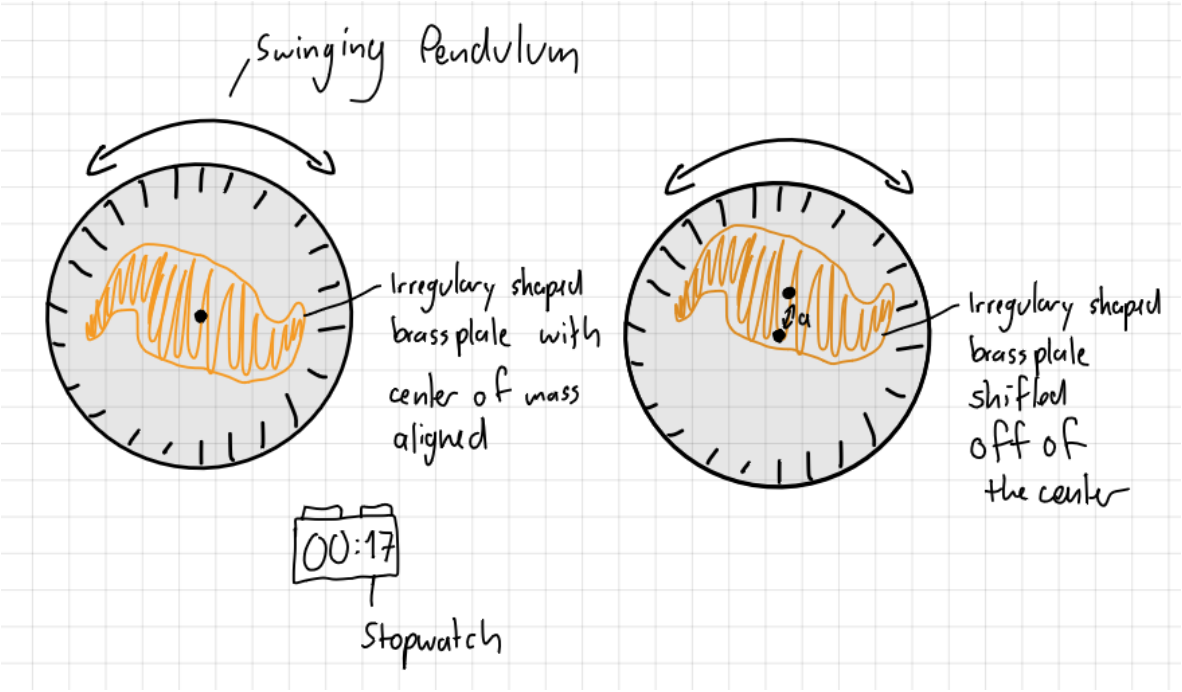


Figure 3: Brass plate on rotary pendulum

Lastly, we take an irregulary shaped brass plate of which we have determined the center of mass by balancing it on a thin beam a few times and noting the line on which it was supported. The intersection gives us the center of mass. We now place it on the rotating table so that the axis of rotation goes through the C.o.M. as seen in figure 3. We measure the period, then move the C.o.M. a few centimeters away from the axis of rotation. We repeat this for five different distances and write down the period and the distance.

1.3 Physical principles

To understand how this experiment works, we need to first take note of a few equations and basic information.
The rotary equations can be described completely analogous to those of linear motion. If you substitute the corresponding quantities, the equations work the same.

- location $r \leftrightarrow$ angle θ
- velocity $\dot{r}$ or $v \leftrightarrow$ angular velocity $\dot{\theta}$ or ω
- mass $m \leftrightarrow$ moment of inertia J
- force $F \leftrightarrow$ torque M
- momentum $p \leftrightarrow$ angular momentum L
- total energy $E_{tot} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \leftrightarrow$ total energy $E_{tot} = \frac{1}{2}D\theta^2 + \frac{1}{2}J\omega^2$

Using this knowledge we can write down the differential equation by either replacing the quantities for the linear differential equation, or by taking the time derivative of the total Energy (which is constant

because of energy conservation).
We get

$$J\dot{\omega} + D\theta = 0 \tag{1}$$

or more traditionally

$$\ddot{\theta} + \omega_0^2\theta = 0 \text{ where } \omega_0 = \sqrt{\frac{D}{J}} \tag{2}$$

By using $T = \frac{2\pi}{\omega_0}$, we get the oscillation period to be

$$T = 2\pi\sqrt{\frac{J}{D}} \tag{3}$$

Another thing we can derive from the equivallencies above is the following:

$$F = ma \leftrightarrow M = J\ddot{\theta} \tag{4}$$

and

$$F = -kr \leftrightarrow M = -D\theta \tag{5}$$

To actually calculate the moment M applied on an object, the formula is as follows:

$$M = F \cdot r \cdot \cos(\varphi) \tag{6}$$

where F is the force applied on the object, r is the distance of that point from the axis of rotation and φ is the angle between the connecting line from axis of rotation to the point where the force is applied and the direction of the force. Since in our example the force is only applied tangentially to a round disk, the angle becomes 90° and the equation simplifies to:

$$M = F \cdot r \tag{7}$$

To calculate the moment of inertial for an object, you have to solve the following integral:

$$J = \int_V \rho(\vec{r}) r^2 \, dV \tag{8}$$

For a disk with uniform mass distribution this returns:

$$J_s = \frac{1}{2} m_s r_s^2 \tag{9}$$

with m_s being the total mass of the disk and r_s being the radius of it.
Moment of inertia is an additive quantity, meaning that you can simply add the moments of two objects in the same system together to get their combined moment:

$$J_{1,2} = J_1 + J_2 \tag{10}$$

Another important property of the moment of inertia ist known as "Steiners Theorem" it states that the moment of inertia for rotation around any axis is equal to the moment of inertia around the parallel axis through the center of mass plus the distance of the axes squared times the mass:

$$J_a = J_{C.o.M.} + ma^2 \tag{11}$$

where $J_{C.o.M.}$ is the moment of inertia for rotation around the parallel axis through the center of mass, m is the mass of the object and a is the distance between the two axes.

Measurement Protocol | Experiment 12 : Moments of Inertia

06.09.24

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Materials:
- Rotary pendulum with angle measure disk
- Brass disk and irregular brass plate
- Stopwatch
- Thin beam

Stats:
Brass plate weight: 695,30 g ± 0,01 g
Brass disk diameter: 10,51 cm ± 0,005 cm
Brass disk weight: 542,13 g ± 0,01 g
Angle measure diameter: 10,01 cm ± 0,005 cm
Angle measure mass: 138,45 g ± 0,01 g

The first step is to calculate the directing torque D of the rotary pendulum. This is done by hanging different weights on a string tangentially connected to the pendulum. The change in angle is noted.

Δm [g]	40	80	120	160	200	240
$\Delta\theta$ [°]	47	95	142	194	242	299

$\Delta_1\theta = \pm 1°$

Δ describes the change, not the inaccuracy

Δ_1 is the symbol we use for inaccuracy

Next we will let the pendulum swing freely for 20 periods and measure the time it takes. We repeat this 3 times:

Measurment number	1	2	3
Time for 20 Periods t [s]	24,46	24,46	24,46

$\Delta_1 t = \pm 0{,}25$

Now we place a brass disk on top of the rotary pendulum and repeat the same measurements as before

Measurment number	1	2	3
Time for 20 Periods t [s]	33,48	33,59	33,43

$\Delta_1 t = \pm 0{,}25$

To calculate the moment of inertia of the irregulary shaped brass plate, we first need to find its center of mass. This is done by placing it on a thin beam and balancing it. Then we mark the line where the beam supported the plane. By doing this multiple times at different angles, we get one point of intersection where the center of mass

Then the plate is placed on top of the rotary pendulum with its C.o.M. in the middle. We repeat the period measurements once more.

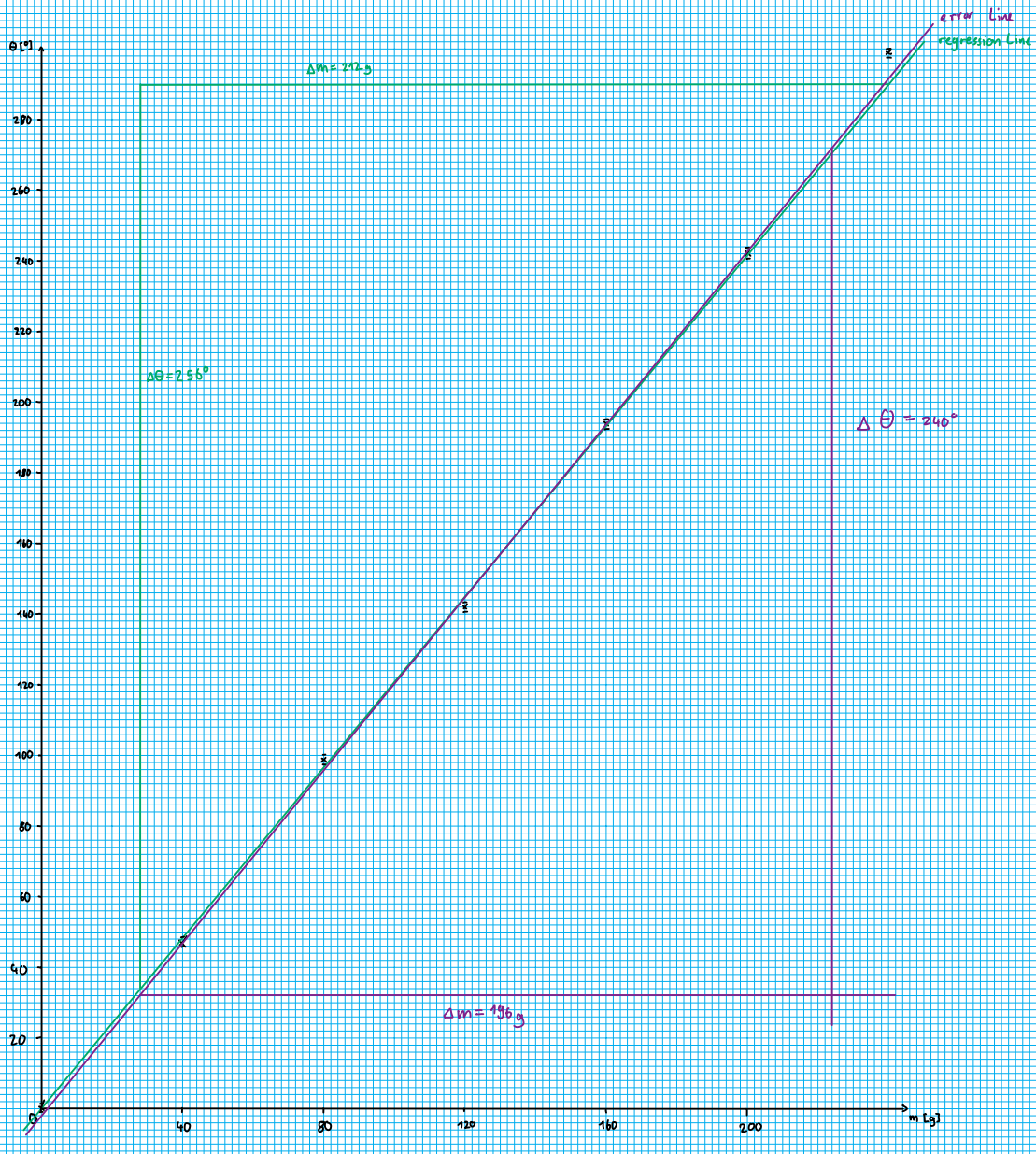
Measurment number	1	2	3
Time for 20 Periods t [s]	47,14	47,23	47,32

$\Delta_1 t = \pm 0{,}25$

Lastly, we measure the period duration for rotational axies parallel to the axis of rotation through the C.o.M with different distances.

Distance a from COM [mm]	5	10	15	20	25
20 Periods t [s]	47,32	48,09	48,58	49,10	50,29

Task 1: Determining the directing moment D using the relationship between the measured Angle θ and the applied Torque M:



Using the regression line, we can determine the gradient $\frac{\Delta\theta}{\Delta m}$ to be $\frac{\Delta\theta}{\Delta m} = \frac{256°}{0{,}212\,kg} = \frac{2\pi \cdot 256}{0{,}212\,kg \cdot 360}$

we know $D = \frac{M}{\theta} = \frac{mgr}{\theta}$ (Formula [?] for reference)

Using the reciprocal of the gradient we can calculate the directing moment to be: $\underline{\underline{D = g \cdot r \cdot \frac{\Delta m}{\Delta \theta}}} = \underline{\underline{23{,}27 \cdot 10^{-3}\,kg\,\frac{m^2}{s^2}}}$ (using the measured radius of the disk of 5 cm)

When doing the same for the error line, we get that:

$$\underline{\underline{D}} = 9{,}81 \cdot 0{,}05\,\frac{m^2}{s^2} \cdot \frac{0{,}196\,kg \cdot 360}{2\pi \cdot 240} = \underline{\underline{22{,}95 \cdot 10^{-3}\,kg\,\frac{m^2}{s^2}}}$$

Therefore we can conclude D to be $\underline{\underline{D = [23{,}27 \pm 0{,}32] \cdot 10^{-3}\,kg\,\frac{m^2}{s^2}}}$

- Note: I assumed the error in mass of the weights and the error to be negligible since the scale showed pretty constant results and the diameter of the plate was measured rather accurately

Task 2: We want to verify our determination of D by calculating it again, this time by a different Method

2.1: Calculating the moment of inertia of the circular brass disc

To calculate the moment of inertia of a disc, we simply use formula [8]

Using it, we get $J_S = \frac{m_S r_S^2}{2}$ (Formula [9])

2.2: Substituting it into the formula of the period duration

Because of formula [3], we know the period duration to be $T = 2\pi\sqrt{\frac{J}{D}}$

Therefore the time we measured without the brass plate is:

$T_1 = 2\pi\sqrt{\frac{J_T}{D}}$ (J_T being the moment of inertia of the spinning table)

and with the brass plate:

$T_2 = 2\pi\sqrt{\frac{J_T + J_S}{D}}$

Squaring both of these relations and subtracting T_1^2 from T_2^2 leaves us with:

$$T_2^2 - T_1^2 = 4\pi^2 \frac{J_T + J_S}{D} - 4\pi^2 \frac{J_T}{D} = 4\pi^2 \frac{J_S}{D}$$

$$\Leftrightarrow \quad D = 4\pi^2 \frac{J_S}{T_2^2 - T_1^2}$$

Now substituting the J_S our new way of calculating D becomes

$$D = 2\pi^2 \frac{m_S r_S^2}{T_2^2 - T_1^2}$$

Our best possible estimate will use the mean of our time measurements divided by 20

Those are as follows: $T_1 = \frac{3 \cdot 24,46}{60} s = 1,22\,s$

$$T_2 = \frac{33,48 + 33,59 + 33,43}{60} s = 1,67 s$$

We find out, that, using this method, $D = 22,56 \cdot 10^{-3}\, kg\, \frac{m^2}{s^2}$

To determine the error of this way of calculating D, we use gaussian error propagation:

$$\Delta D = \sqrt{\left[\frac{\partial D}{\partial m_S} \Delta m_S\right]^2 + \left[\frac{\partial D}{\partial r_S} \Delta r_S\right]^2 + \left[\frac{\partial D}{\partial T_2} \Delta T_2\right]^2 + \left[\frac{\partial D}{\partial T_1} \Delta T_1\right]^2}$$

$$= \left(\left[2\pi^2 \frac{r_S^2}{T_2^2 - T_1^2} \Delta m_S\right]^2 + \left[4\pi^2 \frac{m_S r_S}{T_2^2 - T_1^2} \Delta r_S\right]^2 + \left[4\pi^2 \frac{m_S r_S^2\, T_2}{(T_2^2 - T_1^2)^2} \Delta T_2\right]^2 + \left[4\pi^2 \frac{m_S r_S^2\, T_1}{(T_2^2 - T_1^2)^2} \Delta T_1\right]^2 \right)^{\frac{1}{2}}$$

$$= 1,2 \cdot 10^{-3}\, kg\, \frac{m^2}{s^2}$$

Therefore $\underline{D = (22,6 \pm 1,2) \cdot 10^{-3}\, kg\, \frac{m^2}{s^2}}$ which is not a significant difference from the result obtained above.

Task 3: Calculating the moment of inertia of the irregulary shaped brass disc

To calculate the moment of inertia for the brass plate, we will use a similar approach to what we did in task 3.

We remember that the period duration for the empty table is calculated by $T_1 = 2\pi\sqrt{\frac{J_T}{D}}$ and that the period duration for the table with the plate strapped ontop of it is described as $T_3 = 2\pi\sqrt{\frac{J_T + J_P}{D}}$ (J_P being the moment of inertia of the plate)

Using the same squaring technique as before, we get the following relationship:

$$T_3^2 - T_1^2 = 4\pi^2 \frac{J_P}{D} \iff J_P = D \frac{T_3^2 - T_1^2}{4\pi^2}$$

For T_3 we use $T_3 = \frac{47,14 + 47,23 + 47,32}{60}\,s = 2,36\,s$

ΔT_3 being $\frac{\Delta t}{20} = 0,01\,s$

As D we use the value we found by analyzing the graph since it has the smaller error.

It follows, that $J_P = 2,40 \cdot 10^{-3}\ kg\,m^2$

The error will again be calculated by error propagation.

$$\Delta J_P = \sqrt{\left(\frac{\partial J_P}{\partial D}\Delta D\right)^2 + \left(\frac{\partial J_P}{\partial T_3}\Delta T_3\right)^2 + \left(\frac{\partial J_P}{\partial T_1}\Delta T_1\right)^2}$$

$$= \frac{1}{4\pi^2}\sqrt{\left(T_3^2 - T_1^2\right)^2 \Delta D^2 + \left(2DT_3\,\Delta T_3\right)^2 + \left(2DT_1\,\Delta T_1\right)^2}$$

$$= 0,05 \cdot 10^{-3}\ kg\,m^2$$

We conclude, that the moment of inertia equals: $J_P = (2,40 \pm 0,05) \cdot 10^{-3}\ kgm^2$

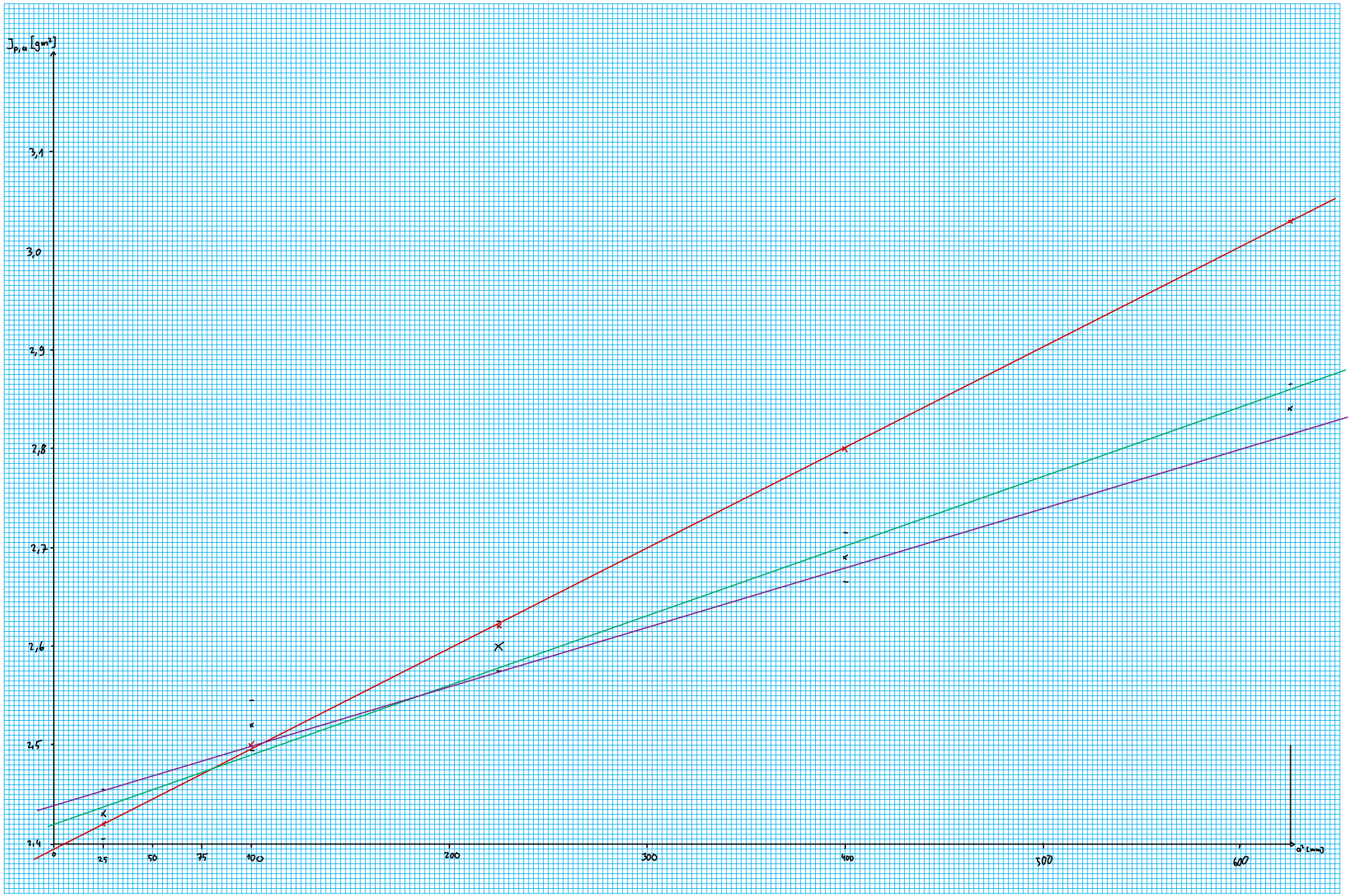
Task 4: Graphing the $J_p - a^2$ relation

Using the formula found in task 3 we can create a table for different distances a between the center of mass of the brass plate and the axis of rotation:

distance a [mm]	5	10	15	20	25
distance a^2 [mm²]	25	100	225	400	625
period measured T_0 [s]	2,37	2,40	2,43	2,46	2,51
resulting moment of inertia $J_{p,a}$ [kg m²]	$2,43 \cdot 10^{-3}$	$2,52 \cdot 10^{-3}$	$2,60 \cdot 10^{-3}$	$2,69 \cdot 10^{-3}$	$2,84 \cdot 10^{-3}$
Error of $J_{p,a}$: $\Delta J_{p,a}$ [kg m²]	$0,05 \cdot 10^{-3}$	$0,05 \cdot 10^{-3}$	$0,05 \cdot 10^{-3}$	$0,05 \cdot 10^{-3}$	$0,05 \cdot 10^{-3}$
$J_{p,a}$ using Steiners Theorem [kg m²] ↑ Formula (11)	$2,42 \cdot 10^{-3}$	$2,50 \cdot 10^{-3}$	$2,62 \cdot 10^{-3}$	$2,80 \cdot 10^{-3}$	$3,03 \cdot 10^{-3}$

Red: Steiners Theorem
Green: regression Line
Purple: error Line

Moments of inertia



4 Summary and Discussion of the results

To summarize the experiment, we first measured the directing moment D of the rotary pendulum through measuring the displacement of the rotary pendulum as a function of torque applied. We then verified this result by determining the directing moment trough the oscillation period of the pendulum with and without a disk with known inertia. Using this quantity we were able to calculate the moment of inertia of a plate with a complex shape by measuring the oscillation period of the pendulum with the plate placed on it with its center of mass aligned to the center of the table, and comparing it to the period of the empty table.
Lastly we tried to verify Steiners Theorem by shifting the plate a few centimeters away from the axis of rotation. We compared these result to the expected period calculated from Steiners Theorem.
When comparing the two directing moments calculated in task one and two, we notice that, using this formula:

$$\text{Difference} = \frac{|D_1 - D_2|}{\sqrt{(\Delta D_1)^2 + (\Delta D_2)^2}} \tag{12}$$

Our difference is about $0,54\sigma$, which is not a significant difference and implies, that the real value of D is rather accurately determined. However, we also see, that the first way of calculating it yields much more accurate results. This is most likely due to the fact, that it is much easier to read an angle on a disk than it is to accurately stop the time. To lower the error of the second method I would probably try to increase the oscillations counted, since this lowers the error of human reaction time.
In the second part of the experiment however, the determined values for different distances did not really align with the expected values of Steiners Theorem, despite the error already being rather big. This could be due to various reasons. For example we could've made an error when counting oscillations or when measuring the distances. Another possible issue could be our determined center of mass of the plate, since we never really got it to balance on the beam and we can't verify our result for its moment of inertia.
The most likely error however is inputting the results in the calculator. Especially the formula for the error is incredibly long, but also the formula for calculating the best possible guess is easy to mess up when typing it into such a tiny screen. I double-checked almost all calculations but I cannot guarantee spotting every mistake.
When reattempting this experiment, I would primarily increase the oscillations measured for determining the moment of inertia (or directing moment), but also use some sort of program to evaluate the results or plot the graphs.
Although the last part of the experiment wasn't a great success, I am still proud of at least the first part, since both calculations came to approximately the same result.